The Effect of Fringing Fields on the Resistance of a Conducting Film

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Abstract **—We have calculated the effect of fringing fields on the measured resistance of a conducting film between two circular disks, using two complementary approaches, for a wide range of disk separations. The problem is cast as the numerical solution of a dual integral equation and a straightforward relaxation procedure for the isomorphic problem of the fringing effects on the capacitance of a circular disk between two grounded planes. These results also represent the solution for the capacitance in the high dielectric limit for two disks separated by a dielectric medium.**

I. Introduction

In ORDER TO measure the resistivity of a thin film in its perpendicular direction, the effects of field spreading on the measurement of resistance must be taken into account. An elementary calculation of resistance from the resistivity ignores the effect of these fringing fields, which lower the resistance in qualitatively the same way as adding external resistors in parallel with the sample. In this work, we calculate these geometrically induced corrections by solving the isomorphic problem of the capacitance of a circular disk placed between two parallel grounded planes. For an infinite, homogeneous, ohmic medium, the problems are connected through the elementary relation

$$RC = \epsilon/\sigma \tag{1}$$

where ϵ is the dielectric constant of the material and σ is the conductivity, and R and C are the resistance and the capacitance, defined as usual. We use two methods to determine the capacitance of a disk of radius one and a normalized separation κ ($=d/2a$, the radius a having been set to one. See inset of Fig. 1), first simply solving LaPlace's equation for the potential at discrete points by relaxation (mean value theorem), and by solving the dual integration equation for this geometry numerically. The LaPlace equation's applicability is seen if we consider the continuity equation

$$\nabla \cdot J = d\rho/dt \tag{2}$$

noting that for steady-state conduction $d\rho/dt = 0$, and using $J = \sigma E$, then

$$\sigma \nabla \cdot E = 0 \tag{3}$$

and so, since $\nabla \times E = 0$ meaning $E = -\nabla\Phi$, Φ some potential

$$\nabla^2 \Phi = 0 \tag{4}$$

which is simply Laplace's equation.

The two approaches are complementary in the sense that the systematic errors in the two methods increase for different limits of κ. Fortuitously, there is also an adequate region of overlap between the results of these methods. These results, in the appropriate limiting cases, are compared to an asymptotic expansion for this problem developed elsewhere [1], and to the results of H. A. Wheeler [2].

II. Methods

A. Method I: The Integral Equation

Sneddon [3] has shown that the solution for the potential of this geometry (see inset Fig. 1) can be given as

$$V(\rho, z) = \int_0^\infty \frac{u^{-1}\sinh(\kappa - |z|)uA(u)J_0(pu)\,du}{\cosh \kappa u} \tag{5}$$

where ρ is the radial coordinate, z is the vertical coordinate, and φ, the angular coordinate, is ignorable because of the circular symmetry of the problem. $A(u)$ must then satisfy the dual integral equations

$$\int_0^\infty u^{-1}\tanh(\kappa u)A(u)J_0(\rho u)\,du = f(\rho), \qquad 0 \le \rho \le 1 \tag{6a}$$

and

$$\int_0^\infty A(u)J_0(\rho u)\,du = 0, \qquad \rho > 1. \tag{6b}$$

In these equations, $J_0(\rho u)$ is the usual first-order Bessel function of the first kind and $f(\rho)$ describes the potential (see [3]) on the disk, which in this case will be taken to be a constant and set to one. Furthermore, if we set

$$A(u) = u\int_0^1 \Phi(t)\cos(ut)\,dt \tag{7}$$

then the integral equations are satisfied if

$$\Phi(t) = h(t) + \int_0^1 \Phi(\tau)Q(t,\tau)\,d\tau \tag{8}$$

where

$$h(t) = \frac{2}{\pi}\frac{d}{dt}\int_0^t \frac{f(\rho)\rho\,d\rho}{\sqrt{(t^2-\rho^2)}} = 2/\pi \quad \text{for } f(\rho) = 1 \tag{9}$$

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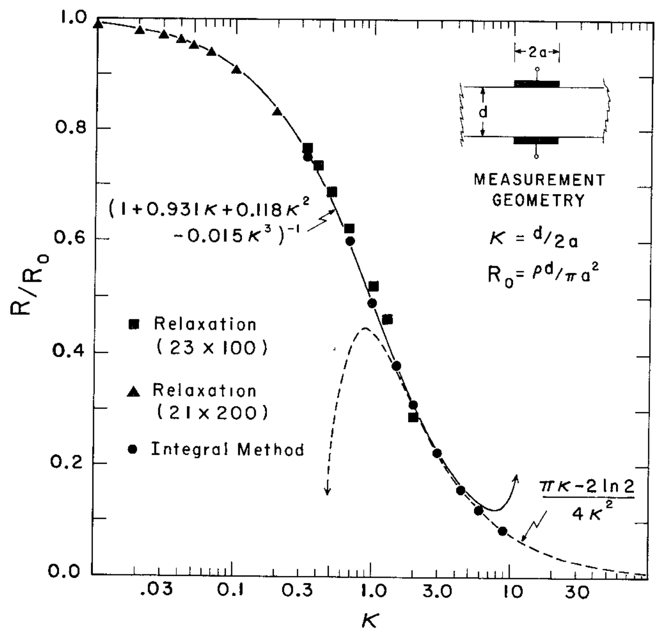

Fig. 1. The measured resistance between two disks separated by a conducting medium, normalized to the resistance without field spreading. ($R_0 = \rho d/a^2\pi$, ρ is the resistivity.) The plate separation, κ, $= d/2a$; the geometry is pictured in the inset. The circles are points directly calculated by solving the integral equation (eq. (8)) in the text, as described by Sneddon [3], numerically; the squares and triangles are points calcuated by a relaxation procedure, as implemented in two separate programs. The first program, a 23×100 cell relaxation, covers the range of $0.3 < \kappa < 2$, and its results are the squares. The second uses a 21×200 matrix, and calculates results for $0.01 < \kappa < 0.3$ (these are the triangles). The dashed lined is an approximation for the resistance at large separations. A third-order polynomial fit was developed to describe the data represented here by the circles, for the particular region $\kappa \approx 1$. It is plotted as the solid line, and shows good agreement with the results for $\kappa < 1$, and is superseded by the large κ asymptotic expression for $\kappa > 2$.

and

$$Q(t,\tau) = \frac{2}{\pi}\int_0^{\infty}(1-\tanh \kappa u)\cos ut \cos u\tau \, du. \qquad (10)$$

It can be shown, as for the similar problem of two free disks [3], by taking the limit of the potential as $\kappa \to \infty$ that the capacitance normalized to the value $C_0 = 1/(4\kappa)$, is given by

$$C/C_0 (= R_0/R) = 2\kappa \int_0^1 \Phi(t)\, dt. \qquad (11)$$

On implementation, a 10×10 or a 64×64 transform matrix $Q(n,m)$ was set up by integrating (10) for values of t and τ ascribed to the discrete variables n and m, using a simple trapezoidal rule. $\Phi(n)$, the discrete analog of $\Phi(t)$, is then iterated as a whole until it satisfies (that is, each point is within 10^{-6} of its value in the previous iteration)

$$\Phi(n) - \frac{2}{\pi} = \frac{1}{N}\sum_{m=1}^{N}\Phi(m)Q(n,m) \qquad (12)$$

where N is the total number of discrete elements n, the discretization of t, assigned to $\Phi(t)$, and m is a discretization of τ. This expression is structurally equivalent to (8).

B. Method II: Relaxation

A relatively straightforward method for determining the capacitance of this system, especially as $\kappa > 0$, is to solve for the potential in a relaxation scheme solving Laplace's equation. Our method follows that of Bartlett and Corle [4], who used this to compare the potential on the plate to the values found by Love [5], Nomura [6], and Cooke [7] for the problem of two free plates. Our problem is relatively more simple, as the boundaries imposed by model considerations are more representative of the boundary conditions of this problem than of theirs. In contrast to their work, which focused on determining the field at each point, we will instead be considering the capacitance of this system, using the result that [8]

$$C = 1/4\pi \int |\nabla \Phi|^2 \, dV \qquad (13)$$

that is, that the capacitance of a disk at unit potential in a grounded box is just the integral over the volume of the gradient squared. Without spreading this reduces to the elementary result $C_0 = \pi a^2/4\pi d$.

In cylindrical coordinates, the problem reduces to a relaxation in two dimensions (φ becomes redundant), with the averaging function modified to average rings at (ρ, z), giving proportional weight to rings at larger radii. Bartlett and Corle [4] have shown that the algorithm for generating the appropriate potential is

$$\begin{aligned} V(i,j) = 1/4\{ & V(i,j^{-1}) + V(i,j+1) \\ & + 1/2[V(i-1,j) + V(i+1,j) \\ & + 1/i((i+1)V(i+1,j) + (i-1)V(i-1,j))]\} \end{aligned} \qquad (14)$$

where $V(i,j)$ is the potential at (i,j), and r is the radius $\propto i$. While this is described as approximating LaPlace's equation to third order, a more rudimentary explanation is that the "volume" of the cell at a radius i is roughly related to its neighbors as $[(i+1)^2 - i^2]$, $[(i^2 - (i-1)^2]$, and their average, representing the volume at $i+1$, $i-1$, and at i, respectively. This leads to

$$\begin{aligned} V(i,j) = 1/8i\,[& 2iV(i,j-1) + 2iV(i,j+1) \\ & + (2i+1)V(i+1,j) + (2i-1)V(i-1,j)]. \end{aligned} \qquad (15)$$

Some simple algebraic manipulation then reproduces their result.

For example, in the $\kappa \sim 1$ case (Section III-C), we used a 23×100 cell matrix $V(i,j)$, "grounding" rows 1 and 23 and putting the "disk," of unit potential and length a, at row 12. We simultaneously iterated, using a separate matrix to store the new values, $V(i,j)$ until the values no longer change (the change in any value from one iteration to the next being less than 10^{-7}, the numerical resolution of this single-precision procedure). The first column was held at values corresponding to a constant gradient, as suggested by the symmetry of the problem. The last column was also held at zero potential; this will introduce errors in the values corresponding to smallest κ, but which are weeded out by checking the penultimate column (the

TABLE I
THE NORMALIZED CAPACITANCE VALUES FROM RELAXATION VERSUS KIRCHOFF'S APPROXIMATION, FOR TWO FREE DISKS

κ	$C/C_0^{(a)}$	$C/C_0^{(b)}$	Δ (%)$^{(c)}$
0.250	1.595	1.575	+1.27
0.182	1.452	1.455	−0.21
0.143	1.375	1.380	−0.36
0.118	1.328	1.327	+0.08
0.111	1.319	1.313	+0.47
0.100	1.296	1.288	+0.62
0.087	1.272	1.258	+1.11
0.067	1.195	1.209	−1.15
0.056	1.175	1.181	−0.51
0.047	1.164	1.166	−0.17

This serves as a check upon the relaxation method. κ is the normalized separation, $d/2a$ from the inset of Fig. 1.
(a) Values directly calculated.
(b) Values from the Kirchoff approximation (eq. (16) in the text).
(c) Percentage difference of the direct values from the approximation.

column next to the wall) for potentials greater than 0.001. This implies that the field spreading is sufficiently contained within the sample space for our desired 1-percent accuracy. Another consideration is the finite thickness of the plate. The capacitance will be driven down relative to the zero-thickness results for larger values of κ. In our studies at this level of accuracy, we find by simply doing this problem with various sizes of spaces that this effect is negligible (<1 percent) for $\kappa < 0.3$. In this same manner, we find that corrections to our basic cell size are even less significant than this for cell sizes in this range.

As a check on this method for determining capacitance, we introduced two plates of opposite voltage in a space of 25×100 cells. The results we have obtained are compared to the Kirchoff approximation

$$\frac{C}{C_0} = 1 + \frac{2}{\pi}\kappa\left[\ln\frac{8\pi}{\kappa} - 1\right] \tag{16}$$

which is an expansion in the limit of small plate separation for the capacitance of two free plates [1] for this related geometry. The results from our program for values of the separation where this method should be accurate (i.e., κ in the range $0.05 < \kappa < 0.2$) for this size of space are listed in Table I. We have reasonably good agreement in this range, especially considering the over-simplified boundary conditions we imposed on this particular problem. Comparison to the more accurate results of other papers on this geometry by Chew and Kong [9], [10] enhance this agreement somewhat, but a much more careful implementation of this particular program would be needed to improve upon their results. This should serve to affirm the accuracy of this method for the more carefully implemented method for our original problem.

III. RESULTS

A. Large Separation: $\kappa \gg 1$

For large values of κ, we appeal to the integral equations of the two methods outlined above, noting in this case that $\Phi(t)$ is very nearly constant, as the planes are far from the disk. As $\kappa \to \infty$, (10) becomes

$$\int_0^\infty (1 - \tanh\kappa u)(1 + O(u^2))\, du \approx \ln 2/\kappa. \tag{17}$$

Since $1 - \tanh ax$ goes to zero quickly if $a \gg 1$, u remains very small over the range where the integrand has an appreciable value, and since t and τ are bounded above by 1.0, we can then replace $\cos ut$ and $\cos u\tau$ by $\cos(0) = 1.0$, leaving terms of order u^2. Therefore, in this limit, $Q(t,\tau)$ is $\ln 2/\kappa$ for all t and τ, and (12) is

$$\frac{\pi}{2}\Phi = \Phi\frac{\ln 2}{\kappa} + 1 \tag{18}$$

which leads to

$$R/R_0 = \frac{(\pi\kappa - 2\ln 2)}{4\kappa^2} \quad \text{as } \kappa \to \infty \quad (\text{note: } R_0 \to \infty \text{ as } \kappa \to \infty) \tag{19}$$

this represents the large κ limit of R/R_0, and is plotted as the dashed line in Fig. 1. Some representative values are given in the fourth column of Table II.

TABLE II
NORMALIZED VALUES OF RESISTANCE R/R_0 OBTAINED THROUGH SOLVING THE INTEGRAL EQUATION (5) AND (8)

	direct calculations		fits to data	
κ	$R/R_0^{(a)}$	$R/R_0^{(b)}$	$R/R_0^{(c)}$	$R/R_0^{(d)}$
9.0	0.0830	0.0830	0.0830	—
6.0	0.1213	0.1213	0.1213	—
4.5	0.1576	0.1576	0.1574	0.1610
3.0	0.2241	0.2242	0.2233	0.2247
2.0	0.3097	0.3103	0.3061	0.3111
1.0	0.4861	0.4916	0.4388	0.4916
0.667	0.5831	0.6000	—	0.5992
0.500	0.6464	—	—	0.6698
0.333	0.6702	0.7502	—	0.7559

κ is the normalized separation. See inset, Fig. 1.
(a) From a 10-place approximation to $\Phi(t)$ (see text under Method I).
(b) From a 64-place approximation to $\Phi(t)$ (see text under Method I).
(c) Calculated from the large κ limit expression

$$\frac{\pi\kappa - 2\ln 2}{4\kappa^2} \quad (\text{see eq. (19)}).$$

(d) Calculated from the polynomial fit $(1 + .931\kappa + .118\kappa^2 - .015\kappa^3)^{-1}$ (see eq. (20)).

B. Small Separation: $\kappa \ll 1$

Based on the previously described relaxation method, another program along these same lines was developed which used a plate "voltage" of 10 and ended up averaging over cells of rectangular rather than square "cross section." This 21×200 program mimicked a much larger space in this way by making the apparent distance between the disk and the planes 2 rather than 10, and so extending this method well into the range where small κ expansions are valid. These values of R/R_0 are listed in Table III, second column, and are the triangles in Fig. 1.

C. Unit Separation: $\kappa \sim 1$

For $\kappa \to 0$, as noted by Chew and Kong [1], $Q(n,m)$ of Method I becomes a diagonal matrix, but for this region,

TABLE III
VALUES OBTAINED FROM THE RELAXATION METHOD, AS ADAPTED FOR SMALL κ (SEE TEXT)

	direct	comparisons	
κ	R/R_0 (a)	R/R_0 (b)	R/R_0 (c)
0.200	0.8320	0.8389	0.8429
0.100	0.9104	0.9133	0.9162
0.067	0.9382	0.9413	0.9429
0.050	0.9531	0.9550	0.9566
0.040	0.9623	0.9637	0.9651
0.033	0.9684	0.9699	0.9711
0.029	0.9729	0.9736	0.9745
0.025	0.9762	0.9772	0.9779
0.022	0.9788	0.9799	0.9806
0.020	0.9809	0.9816	0.9823
0.018	0.9826	0.9833	0.9841
0.015	0.9852	0.9858	0.9867
0.013	0.9872	0.9878	0.9884
0.012	0.9887	0.9889	0.9893

The polynomial is taken from Chew and Kong [1] and tabulated here for comparison to these values in this regime.

(a) Values directly calculated by relaxation, with the modifications described in the section on $\kappa<1$, allowing determination of the resistance for these values of κ.

(b) Calculated from the polynomial fit to the data of Table II, $(1+0.931\kappa+0.118\kappa^2-0.015\kappa^3)^{-1}$ (see eq. (20)).

(c) Calculated from the polynomial derived from the results of Chew and Kong [1], $(1+0.898\kappa+0.171\kappa^2)^{-1}$ (see eq. (23)).

TABLE IV
VALUES OF R/R_0 FROM RELAXATION METHOD

κ	R/R_0 (a)	R/R_0 (b)
2.0	0.3482	0.3111
1.67	0.3920	0.3559
1.25	0.4640	0.4312
1.00	0.5211	0.4916
0.667	0.6250	0.5992
0.500	0.6901	0.6698
0.455	0.7107	0.6916
0.400	0.7369	0.7194
0.345	0.7657	0.7491
0.333	0.7722	0.7559
0.303	0.7886	0.7734
0.286	0.7988	0.7837

(a) Direct calculations from the relaxation procedure described under Method II in the text.

(b) Calculated from the polynomial fit to the data listed in Table II (see eq. (20)).

$\kappa\sim 1$, the off-diagonal elements can become very important. In consideration of this, we note that $Q(t,\tau)$ is always nonnegative, and tends to zero as $\kappa\to 0$ for $t\neq\tau$. Also, the average of each row or column in $Q(i,j)$, or the integral on either t or τ in $Q(t,\tau)$ must be less than $\pi/2$, or a solution of (8) will not exist. The limitations of this method, for $\kappa<1$, stem from the need to divide $\Phi(t)$ into many places (N must be large) so that this condition on the average element in any row of $\Phi(n)$ is met. Our crude integration scheme tends to overestimate $Q(i,j)$, and for N too small will lead to vast overestimates of C/C_0 (underestimating R/R_0). Hence, we did not use the integral equation to generate results for Section III-B. For our purposes, we have found that, by using $N=64$, we have sufficient agreement between the two methods used (the intrinsic numerical errors tend to increase for opposing limits of κ, in comparing the integral method to relaxation) to have confidence in our calculations to the 1-percent level. Data obtained from this method for R/R_0 is given in Table I, and their inverses R/R_0 are plotted as circles in Fig. 1. This method becomes impractical for $\kappa<1/3$ due to excessively slow convergence.

Using the relaxation procedure discussed in Section II, we calculated the resistance of this system for the range of separation $0.28<\kappa<2.0$. The numbers obtained in this manner are presented in Table IV (second column), and are represented as squares in Fig. 1.

IV. COMPARISONS OF RESULTS

In order to compare our results calculated for separation $\kappa\sim 1$ to those obtained for small plate separation [1], we fit our capacitance results to a polynomial in the plate separation. A third-order chi-squares fit to the results (for the range $1/3<\kappa<3$) generated by the $N=64$ place program gave the polynomial

$$C/C_0=1+0.931\kappa+0.118\kappa^2-0.015\kappa^3 \tag{20}$$

which we will use to compare to other results. Its inverse, being the normalized resistance R/R_0, is the solid line in Fig. 1, with some values tabulated in column 5 of Table II and column 3 of Tables III and IV.

Chew and Kong [1] (here as CK) solved the equation for $\Phi(t)$ (as in Section I) in the limit of $\kappa\ll 1$ where the function $Q(t,\tau)$ is diagonal and where the equations simplify, to obtain an expansion for the capacitance of two disks separated by a moderate dielectric. Their full expression, which involves two circular disks in free space (or a host medium) separated by a dielectric medium, is in our notation (see inset, Fig. 1 for geometry)

$$C\approx\frac{a^2\pi\epsilon_1\epsilon_0}{4\pi d}\left\{1+\frac{2d}{\pi\epsilon_1 a}\left\{\ln\left(\frac{a}{2d}\right)+(1.41\epsilon_1+1.77)+\frac{d}{a}(0.268\epsilon_1+1.65)\right\}\right\} \tag{21}$$

where ϵ_0 is the host dielectric constant and ϵ_1 is the dielectric constant of the medium separating the plates. This becomes, normalized to $C_0=a^2\pi/4\pi d$, for $\epsilon_1\to\infty$

$$C/C_0\approx\lim_{\epsilon_1\to\infty}\left\{1+\frac{2d}{\pi\epsilon_1 a}\left\{\ln\frac{a}{2d}+\left(1.41+0.268\frac{d}{a}\right)\epsilon_1\right\}\right\} \tag{22}$$

$$\approx 1+0.898\kappa+0.171\kappa^2. \tag{23}$$

These values of C/C_0 are also listed in Table II(b). These results do not differ greatly from those obtained by the third-order fit developed from points taken from the solution of this problem for an entirely different set of plate separations. For the range claimed accurate by CK ($\kappa<1/2$), these values differ by at most 2 percent. (Some values are given for comparison in column 4 of Table III). The usefulness of the third-order fit, in contrast to this second-order expression, is to accurately describe the capacitance in the range $\kappa\sim 1$, where the extra term enables us to then switch over to the expression for $\kappa \gg 1$ (eq. (16)) at approximately $\kappa \approx 3$ while remaining within ~ 1 percent of the values directly calculated. In the graph of R/R_0 ($\equiv C_0/C$, Fig. 1) the line for (23) would be indistinguishable from the fit (eq. (17)) for $\kappa < 0.2$ and diverge only slightly (on the low side of the calculated resistance) after that. Over the rest of the range of separations, we compare to the results of Wheeler [2], who analytically interpolated the asymptotic expressions for the large and small separation limits. There is excellent agreement everywhere except for the range $0.02 < \kappa < 0.4$, where this expression

$$\frac{8\kappa + 2\pi}{2\pi}\left\{1-(4+1.3k^{-1}+5.8\kappa)^{-1}\right\} \tag{24}$$

over-estimates the actual resistance by more than 1 percent.

V. Conclusions

We have determined the resistance between two circular, coaxial disks on opposing sides of a thin slab of conducting medium in the presence of fringing fields for disk separations of 0.01 to 10.0 times the disk diameter. We used two complementary procedures to solve the equivalent problem of the capacitance of a single disk between two grounded planes: one a straightforward relaxation solving LaPlace's equation on a computer grid, another a numerical solution of the derived integral equation appropriate to this system. As a check upon our relaxation method, we also calculated the capacitance of two free disks, and find very good agreement with the Kirchoff approximation for a relatively large range of disk separation.

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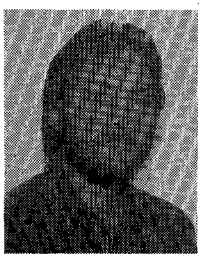

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